Spring 2007 Hutchings Final

7. Consider the vector field

$$F = \sqrt{x^2+y^2+z^2}\langle x, y, z\rangle.$$

(a) There is a constant c such that $\operatorname{div} F = c\sqrt{x^2+y^2+z^2}$. Find c.

(b) Compute the outward flux of the vector field F through the boundary of the solid region $x^2+y^2+z^2 \le 1$, $z \ge \sqrt{x^2+y^2}$

(a) $F = \langle \underbrace{x\sqrt{x^2+y^2+z^2}}_{P}, \underbrace{y\sqrt{x^2+y^2+z^2}}_{Q}, \underbrace{z\sqrt{x^2+y^2+z^2}}_{R} \rangle$

$$\operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\frac{\partial P}{\partial x} = (1)(x^2+y^2+z^2)^{\frac{1}{2}} + (x)(\tfrac{1}{2})(x^2+y^2+z^2)^{-\frac{1}{2}} \cdot 2x$$

$$= \sqrt{x^2+y^2+z^2} + \frac{\cancel{2}x^2}{\cancel{2}\sqrt{x^2+y^2+z^2}}$$

$$= \frac{(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}} + \frac{x^2}{\sqrt{x^2+y^2+z^2}} = \frac{2x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial Q}{\partial x} = (1)(x^2+y^2+z^2)^{\frac{1}{2}} + (y)(\tfrac{1}{\cancel{2}})(x^2+y^2+z^2)^{-\frac{1}{2}} \cdot \cancel{2}y$$

$$= \sqrt{x^2+y^2+z^2} + \frac{y^2}{\sqrt{x^2+y^2+z^2}} = \frac{x^2+2y^2+z^2}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial R}{\partial z} = (1)(x^2+y^2+z^2)^{\frac{1}{2}} + (z)(\tfrac{1}{\cancel{2}})(x^2+y^2+z^2)^{-1/2} \cdot \cancel{2}z$$

$$= \sqrt{x^2+y^2+z^2} + \frac{z^2}{\sqrt{x^2+y^2+z^2}} = \frac{x^2+y^2+2z^2}{\sqrt{x^2+y^2+z^2}}$$

$$\operatorname{div} F = \frac{2x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}} + \frac{x^2+2y^2+z^2}{\sqrt{x^2+y^2+z^2}} + \frac{x^2+y^2+2z^2}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{4(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}} = 4\sqrt{x^2+y^2+z^2}$$

$$\boxed{c = 4}$$

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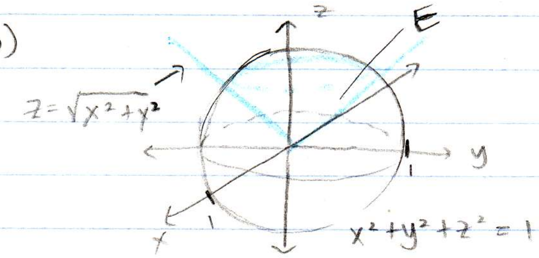
(b)



Flux $= \iint_S F \cdot dS = \iiint_E (\operatorname{div} \vec{F})\, dV$

from (a), we have $\operatorname{div} F = 4\sqrt{x^2+y^2+z^2}$

converting to spherical coordinates,

$\operatorname{div} F = 4\sqrt{\rho^2} = 4\rho$

$x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho\cos\phi$

for the cone: $\rho\cos\phi = \sqrt{\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta}$

$\rho\cos\phi = \sqrt{\rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta)}$

$\rho\cos\phi = \rho\sin\phi$

$\cos\phi = \sin\phi$

$\phi = \pi/4$

$0 \le \phi \le \pi/4 \qquad 0 \le \theta \le 2\pi, \qquad 0 \le \rho \le 1$

$$\int_0^{\pi/4}\int_0^{2\pi}\int_0^1 4\rho \cdot \underbrace{\rho^2 \sin\phi}_{\text{magnification factor}}\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi/4} \sin\phi\, d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^1 4\rho^3\, d\rho$$

$$= -\cos\phi\Big]_0^{\pi/4} \cdot \theta\Big]_0^{2\pi} \cdot \rho^4\Big]_0^1$$

$$= \left[-\cos\left(\tfrac{\pi}{4}\right) + \cos(0)\right] \cdot [2\pi - 0] \cdot [(1)^4 - (0)^4]$$

$$= \left[-\tfrac{\sqrt{2}}{2} + 1\right] \cdot [2\pi] \cdot [1]$$

$$= \boxed{2\pi\left(1 - \tfrac{\sqrt{2}}{2}\right)}$$